Nonlinear Transverse Vibrations of Orthotropic Cylindrical Shells

J. L. Nowinski*
University of Delaware, Newark, Del.

Nonlinear transverse vibrations of elastic orthotropic shells are investigated using von Kármán-Tsien equations generalized to dynamic and orthotropic case. The deflection function is chosen in a simple separable form and the stress function is determined from the compatibility equation. The governing equation for the time function is derived by Galerkin's procedure, and its solution discussed for two types of orthotropy and for the isotropic case. A sharp decrease of the period of nonlinear vibrations with an increasing amplitude is corroborated, the mode pattern influencing the period more than the degree of anisotropy. Parenthetically, the influence of anisotropy on free linear vibrations and on the buckling under normal pressure is discussed.

Nomenclature

x,y	= axial and circumferential coordinates
u,v,w	= displacements of the median surface in the axial, circumferential, and transverse directions
t	= time
$\epsilon_x,\epsilon_y,\gamma_{xy}$	= components of strain in the median surface
$\sigma_x,\sigma_y,\tau_{xy}$	= components of stress in the median surface
E_1,E_2,G_{12}	= Young's moduli in the axial and circumferential directions, and shear modulus
ν_1,ν_2	= Poisson's ratios
k^2,m^2,p^2	= defined by Eq. (1.4)
h,a,l	= thickness, radius, and length of the shell
q_0	= lateral normal pressure
Φ	= stress function [see Eqs. (1.1)]
ρ	= mass density of the wall of the shell
i,j	= numbers of longitudinal half-waves and circumferential waves
α_i	= $i\pi/l$
β_j	= j/a
$f(t),f_0(t)$	= defined by Eq. (1.10)
$\delta_1,\delta_2,\delta_3,\delta^*$	= defined by Eq. (1.12)
$\epsilon_1,\epsilon_2,\delta$	= defined by Eq. (1.18)
ω	= frequency of linear free vibrations
Λ	= defined by Eq. (3.2)
$A,\tau(t)$	= defined by Eq. (4.2)
Λ_1,Λ_3	= defined by Eq. (4.4)
$K(k^*)$	= complete elliptic integral of the first kind
k^*,ω^*	= defined by Eq. (4.5.1)
$T,T^* = 4K/\omega^*$	= periods of linear and nonlinear free vibrations

Introduction

In June 1962, the author of the present note submitted for publication in the Journal of Aerospace Sciences an article on transverse nonlinear vibrations of cylindrical orthotropic shells prepared earlier as a technical report for the University of Delaware.[1] At this opportunity, one of the reviewers of the article drew the author's attention to a paper by Chu published in the October 1961 issue of the Journal of Aerospace Sciences.[2] In this paper, a problem similar to that mentioned was solved independently, confined, however, to the isotropic material. Again the field equations derived in Ref. 2 are similar to those derived in Ref. 1 if one specifies the latter to the isotropic case. Since the contribution of Chu represents a particular case of the solution given in Ref. 1, the present author was glad to note a close agreement between the final results obtained in both papers for the isotropic case. However, a publication of the full text of Ref. 1, after the previous publication of Ref. 2, did not seem necessary. In this connection, the writer has prepared the present note as an abstract of Ref. 1, quoting only those results that may possess a self-dependent interest. To make the note self-contained, the author starts with derivation of the general field equations, using, in contrast to Ref. 2 (in which these equations are derived as Euler-Lagrange equations from Hamilton's principle), the balance of momenta and the compatibility condition. As another difference, the derivation of the crucial differential time equation is carried on using Galerkin's method.

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* Professor of Mechanical Engineering.

1. Fundamentals

Consider a thin circular shell with the dimensions shown in Fig. 1. Locate the origin of curvilinear coordinates x, y at an arbitrary point of one edge of the shell, and measure x in the axial and y in the circumferential direction in the median surface of the undeformed cylinder (of mean radius a). Assume that the wall of the shell is made of an orthotropic material, the principal elastic directions of which coincide with the axes x and y. The external (lateral) face of the shell is subjected to a normal pressure q (x, y), which, for definiteness, is assumed to be constant and equal to q_0. Transverse vibrations of the wall of the shell are supposed to be large, that is, of the order of magnitude of the thickness of the plate. In problems such as this in which vibrations take place principally in the direction of the least stiffness (perpendicular to the median surface), it is reasonable and customary to neglect the inertia terms inherent with the motion in the median surface. One then may represent the stress components in terms of a stress function $\Phi(x,y,t)$ in the form

$$\sigma_x = \partial^2\phi/\partial y^2 \qquad \sigma_y = \partial^2\phi/\partial x^2 \qquad \tau_{xy} = -\partial^2\phi/\partial x\partial y \tag{1.1}$$

thus automatically satisfying the equations of equilibrium in the planes tangent to the median surface of the shell.

Following von Kármán and Tsien,[3] the strain components, including terms up to the second order, are expressed in the following form:

$$\begin{aligned}\epsilon_x &= (\partial u/\partial x) + \tfrac{1}{2}(\partial w/\partial x)^2\\ \epsilon_y &= (\partial v/\partial y) + \tfrac{1}{2}(\partial w/\partial y)^2 - (w/a)\\ \gamma_{xy} &= (\partial u/\partial y) + (\partial v/\partial x) + (\partial w/\partial x)(\partial w/\partial y)\end{aligned} \tag{1.2}$$

where u, v, w are the components of displacement in the x, y, and transverse directions, respectively.

The stress and strain components in the median surface of the orthotropic wall of the shell then are related to each other by the familiar equations

$$\begin{aligned}\epsilon_x &= (\sigma_x/E_1) - \nu_2(\sigma_y/E_2)\\ \epsilon_y &= (\sigma_y/E_2) - \nu_1(\sigma_x/E_1)\\ \gamma_{xy} &= \tau_{xy}/G_{12}\end{aligned} \tag{1.3}$$

where E_1, E_2, and G_{12} denote Young's moduli in the x and y

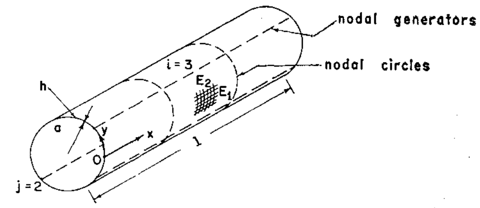

Fig. 1 Dimensions of the shell

directions and the shear modulus, respectively; ν_1 represents the relative contraction in the y direction influenced by the tension in the x direction. Apparently, the relation $E_1\nu_2 = E_2\nu_1$ holds.

In what follows, the following symbolism is used:

$$\begin{aligned} k^2 &= E_2/E_1 = D_2/D_1 \\ p^2 &= 2(G_{12}/E_1)(1 - k^2\nu_1^2) + \nu_2 = D_3/D_1 \\ m^2 &= (E_2/G_{12}) - 2\nu_2 \end{aligned} \qquad (1.4)$$

with

$$D_1 = E_1I/(1 - \nu_1\nu_2) \qquad D_2 = E_2I/(1 - \nu_1\nu_2) \qquad D_3 = \tfrac{1}{2}(D_1\nu_2 + D_2\nu_1) + 2D_k \qquad (1.5)$$

and $I = h^3/12$, $D_k = G_{12}I$, where D_1, D_2, and D_3 represent flexural and torsional rigidities of the wall of the shell, respectively.

The equation of motion in the direction normal to the median surface now yields

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \sigma_x h \frac{\partial^2 w}{\partial x^2} + \sigma_y h\left(\frac{1}{a} + \frac{\partial^2 w}{\partial y^2}\right) + 2\tau_{xy}h\frac{\partial^2 w}{\partial x \partial y} + q_0 - \rho h \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1.6)$$

where ρ denotes the mass density of the material of the wall, t the time, and

$$\begin{aligned} Q_x &= -(\partial/\partial x)[D_1(\partial^2 w/\partial x^2) + D_3(\partial^2 w/\partial y^2)] \\ Q_y &= -(\partial/\partial y)[D_3(\partial^2 w/\partial x^2) + D_2(\partial^2 w/\partial y^2)] \end{aligned} \qquad (1.7)$$

are the shear forces in the transverse and axial cross sections. The variables u and v are now eliminated from Eqs. (1.1–1.3), thus establishing the following compatibility equation:

$$\frac{\partial^4 \phi}{\partial x^4} + m^2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + k^2\frac{\partial^4 \phi}{\partial y^4} = E_2\left[\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \frac{1}{a}\frac{\partial^2 w}{\partial x^2}\right] \qquad (1.8)$$

first obtained by Donnell for the isotropic case. On the other hand, by virtue of (1.1) and (1.7), Eq. (1.6) yields

$$\frac{\partial^4 w}{\partial x^4} + 2p^2\frac{\partial^4 w}{\partial x^2 \partial y^2} + k^2\frac{\partial^4 w}{\partial y^4} = \frac{h}{D_1}\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\frac{\partial^2 \phi}{\partial x^2} - 2\frac{\partial^2 \phi}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y} + \frac{1}{a}\frac{\partial^2 \phi}{\partial x^2} + \frac{q_0}{h} - \rho\frac{\partial^2 w}{\partial t^2}\right] \qquad (1.9)$$

Equation (1.9) represents the second fundamental field equation, besides (1.8), governing the nonlinear dynamical problem for the circularly cylindrical orthotropic shell.

The deflection function $w\ (x, y, t)$ is chosen in the separable form

$$w(x,y,t) = f(t)\sin\alpha_i x \cdot \sin\beta_j y + f_0(t) \qquad (1.10)$$

where $\alpha_i = i\pi/l$ and $\beta_j = j/a$. Clearly, i represents the number of longitudinal half-waves and j the number of circumferential waves. If $i = 0$ or $j = 0$, a uniform radial oscillation is excited which, however, requires that the ends of the cylinder may "breath" without any hindrance. If $j = 1$, a translational motion of the cross sections, with no transverse distortion, occurs. To define a particular vibration pattern, it is necessary to prescribe the values both of i and of j (cf., Fig. 1, in which $i = 3$ and $j = 2$ have been chosen). Expression (1.10) is inserted into the compatibility equation (1.8), and a particular integral is obtained:

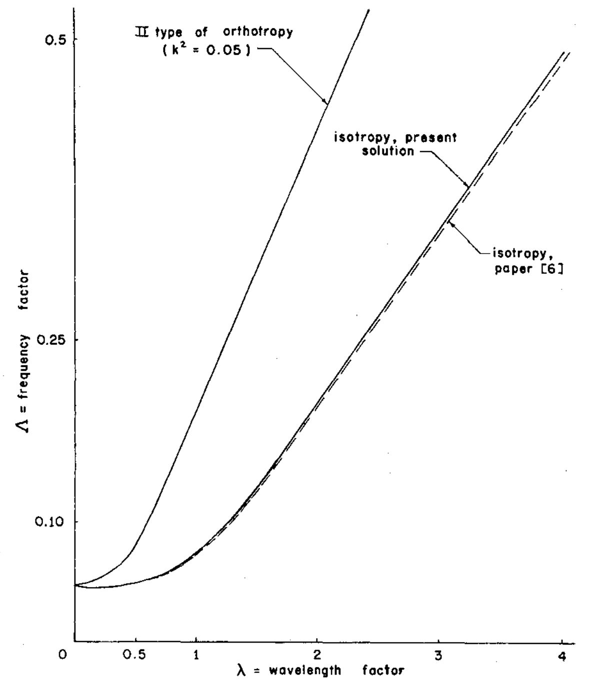

Fig. 2 Frequency factor vs wavelength factor for linear vibrations (j = 4, h/a = 0.01)

$$\Phi(x,y,t) = E_2[(\delta_1\cos 2\alpha_i x + \delta_2\cos 2\beta_j y)f^2 + \delta_3\sin\alpha_i x \cdot \sin\beta_j y] - p_y(x^2/2) \qquad (1.11)$$

where the following notation is used:

$$\delta_1 = \frac{\beta_j^2}{32\alpha_i^2} \qquad \delta_{21} = \frac{\alpha_i^2}{32\beta_j^2k^2} \qquad \delta_3 = \frac{\alpha_i^2}{a\delta^*}$$

$$\delta^* = \alpha_i^4 + m^2\alpha_i^2\beta_j^2 + k^2\beta_j^4 \qquad (1.12)$$

p_y is a constant equal to q_0a/h. The conditions that satisfy the particular solution (1.11) are now investigated.

Note first that in view of (1.11), Eqs. (1.1) for the stress components in the median surface admit the representation

$$\begin{aligned} \sigma_x &= -E_2\beta_j^2(4\delta_2 f^2\cos^2\beta_j y + \delta_3 f\sin\alpha_i x \cdot \sin\beta_j y) \\ \sigma_y &= -E_2\alpha_i^2(4\delta_1 f^2\cos 2\alpha_i x + \delta_3 f\sin\alpha_i x \cdot \sin\beta_j y) - p_y \\ \tau_{xy} &= -E_2\alpha_i\beta_j\delta_3 f\cos\alpha_i x \cdot \cos\beta_j y \end{aligned} \qquad (1.13)$$

With these in mind, it can be shown that the axial resultant force of the normal stresses in any cross section of the shell vanishes. Since this result includes the ends of the shell, it follows that the self-equilibrated systems of axial normal stresses acting at the ends of the shell are associated with what one can interpret as an integral condition for the freely movable edges (in the axial direction). On the other hand, at the end cross sections of the shell, the following statical quantities vanish: 1) the transverse resultant force, 2) an overall twisting moment of the shear stresses, and 3) a resultant bending moment of the normal stresses with respect to an arbitrary diameter. It follows that the reactions at the ends of the shell conform to the equilibrium requirements associated with the prescribed uniform loading of the outer surface of the shell. Furthermore, an inspection reveals that in view of (1.10) the local bending moments at the edges of shell vanish identically. This result in combination with (1.10) conforms to the conditions of the free support of the edges of the shell in an integral sense. Note that the equilibrium of a portion of the shell singled out by an axial cross

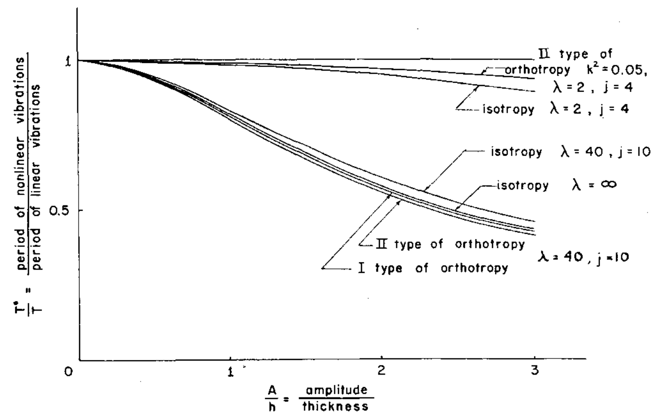

Fig. 3 Relative period vs relative amplitude for nonlinear vibrations

section $y = 0$ (or $y = \pi a$) yields the required equation for the membrane hoop stress:

$$p_y = q_0 a/h \tag{1.14}$$

induced by a uniform lateral pressure q_0.

The next task is to determine the time function $f_0(t)$ in terms of $f\,(t)$ by using the condition of closure (or periodicity of the circumferential displacement)

$$\int_0^{2\pi a} \frac{\partial v}{\partial y}\, dy = 0 \tag{1.15}$$

In view of (1.1–1.3), the desired relation appears as

$$f_0(t) = (\beta_j{}^2 a/8) f^2(t) + (q_0 a^2/hE_2) \tag{1.16}$$

The procedure of Galerkin is now applied to the only remaining equation of motion (1.9). To this end, one proceeds in the following way. Substitute expressions (1.10) and (1.13), representing in fact (with an appropriate sign) the second space derivatives of the stress function Φ, into Eq. (1.9). Then multiply both members of this equation by the spatial part $\sin \alpha_i x \cdot \sin \beta_j y$ of (1.10), and integrate the result over the domain of the median surface of the shell.

As a final result of a lengthy calculation, the following nonlinear differential equation of the second order is obtained for the time function:

$$(d^2f/dt^2) + \epsilon_1 f + \epsilon_2 f^3 = 0 \tag{1.17}$$

where the following notation is used:

$$\begin{aligned} \epsilon_1 &= \frac{D_1}{\rho h}\,\delta + \frac{E_2}{\rho}\left(\frac{\alpha_i{}^2}{a}\,\delta_3 + \frac{\beta_j{}^2 a}{E_2 h}\, q_0\right) \\ \epsilon_2 &= 2\alpha_i{}^2\beta_j{}^2(\delta_1 + \delta_2)(E_2/\rho) \\ \delta &= \alpha_i{}^4 + 2p^2\alpha_i{}^2\beta_j{}^2 + k^2\beta_j{}^4 \end{aligned} \tag{1.18}$$

This completes the general solution of the problem under consideration by reducing it to the solution of the governing time equation (1.17). The types of materials shown in Table 1 are now discussed.

2. Buckling under Uniform Pressure

Rejecting the inertia and nonlinear terms in (1.17), the equation for the uniform transverse pressure q_0 which provokes the buckling of the shell is thus obtained:

$$\frac{\delta h^2}{12k^2(1 - \nu_1\nu_2)} + \frac{\alpha_i{}^4}{\delta^* a^2} = \frac{a\beta_j{}^2}{E_1 h}\, q_0 \tag{2.1}$$

This equation, if specified to the isotropic case, confirms the result obtained in Ref. 4 and represents an orthotropic counterpart of the known equation of Mises, e.g., in Ref. 5.

3. Free Linear Vibrations

Rejecting the nonlinear term in Eq. (1.17), the equation is then reduced to the equation governing linear vibrations. If, moreover, the loading term is suppressed by posing $q_0 = 0$, the equation of free linear oscillations with the circular frequency is obtained:

$$\omega = (1/a)[E_2/\rho(1 - k^2\nu_1{}^2)]^{1/2}\Lambda \tag{3.1}$$

where

$$\Lambda = \left[\frac{\alpha^2}{12k^2}(\lambda^4 + 2p^2\lambda^2 j^2 + k^2 j^4) + \frac{(1 - k^2\nu_1{}^2)\lambda^4}{\lambda^4 + m^2\lambda^2 j^2 + k^2 j^4}\right]^{1/2}$$

is the frequency factor and $\lambda = i\pi a/l$, the wavelength factor introduced in Ref. 6. This result has been plotted in Fig. 2 for the second type of orthotropy and for the isotropic material (with $j = 4$ and $h/a = 0.01$) revealing, in the latter case, a close agreement with the corresponding results of Arnold and Warburton.[6] In the present case, however, for each chosen nodal pattern only one natural frequency (the lowest) exists instead of three possible, although fortunately the only one that appears to have practical significance (the others being well above the aural range). This fact is, apparently, the consequence of neglecting the inertia associated with inplane motion. The ratio of dimensions chosen and $j = 4$ seem to result in a "primarily radial" mode of minimum frequency (see Ref. 7).

4. Free Nonlinear Vibrations

Turn now to the investigation of large vibrations of a shell to which is attributed one degree of freedom associated with the parametric function $f(t)$. To this purpose, a solution $f(t)$ of the complete time equation (1.17) has to be found. It is expedient to introduce the following representation:

$$f\,(t) = A\tau(t) \tag{4.1}$$

which permits the use of the normalized initial conditions:

$$\tau(0) = 1 \qquad \dot{\tau}(0) = 0 \tag{4.2}$$

Upon restriction to free vibrations, the loading term is suppressed in (1.18). With this in mind, Eq. (1.17) now is carried into

$$(\partial^2\tau/\partial t^2) + \Lambda_1\tau + \Lambda_3(A/h)^2\tau^3 = 0 \tag{4.3}$$

where the following notation is used:

$$\begin{aligned} \Lambda_1 &= \frac{E_2}{\rho a^2}\left[\frac{\alpha^2(\lambda^4 + 2j^2p^2\lambda^2 + k^2j^4)}{12(1 - \nu_1\nu_2)k^2} + \frac{\lambda^4}{\lambda^4 + m^2j^2\lambda^2 + k^2j^4}\right] \\ \Lambda_3 &= \frac{E_2}{\rho a^2}\,\alpha^2\,\frac{\lambda^4 + j^4k^2}{16k^2} \end{aligned} \tag{4.4}$$

Table 1

Type	E_1	E_2	G_{12}	ν_1	ν_2	k^2	p^2	m^2
Orthotropy I	1×10^3	0.5×10^5	0.1×10^5	0.05	0.025	0.5	0.223	5
Orthotropy II	1×10^5	0.05×10^5	0.05×10^5	0.2	0.01	0.05	0.108	1
Isotropy	1×10^5	1×10^5	G	0.3	0.3	1	1	2

Obviously, $\Lambda_1 = \omega^2$, where ω is defined by (3.1).

A solution of the foregoing equation satisfying the initial conditions (4.2) is the cosine-type Jacobian elliptic function

$$\tau(t) = cn(\omega^* t, k^*) \tag{4.5}$$

with

$$\omega^* = [\Lambda_1 + \Lambda_3(A/h)^2]^{1/2}$$
$$k^{*2} = \frac{\Lambda_3(A/h)^2}{2[\Lambda_1 + \Lambda_3(A/h)^2]} \tag{4.5.1}$$

Since the period of the linear vibrations is $T = 2\pi/\Lambda_1^{1/2}$ the ratio of periods of nonlinear and linear vibrations is obtained as

$$\frac{T^*}{T} = \frac{2K}{\pi[1 + (\Lambda_3/\Lambda_1)(A/h)^2]^{1/2}} \tag{4.6}$$

In Fig. 3, the relation (4.6) is plotted for two types of orthotropy discussed previously and, for the sake of comparison, also for the isotropic case. For definiteness, it is assumed $\alpha = h/a = 0.01$, and posed $j = 10$ and $\lambda = 40$. An inspection reveals a known sharp decrease of the period of vibrations with an increasing amplitude. Also, graphs for isotropic case and for the II type of orthotropy are plotted assuming $j = 4$ and $\lambda = 2$. Apparently, the mode patterns influence the period of the nonlinear vibrations more than the degree of anisotropy (at least for the two types of orthotropy considered).

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Flutter of Thin Plates under Combined Shear and Normal Edge Forces

J. G. Eisley*
University of Michigan, Ann Arbor, Mich.
AND
G. Luessen†
The Boeing Company, Wichita, Kan.

Stability boundaries are obtained for several cases of simply supported rectangular panels as follows: unbuckled plates subjected to in-plate shear, unbuckled plates under in-plane shear and normal edge forces, unbuckled plates with sweepback, and panels buckled by equal compressive stresses in both the spanwise and chordwise directions. For the unbuckled case, small deflection thin plate theory and Galerkin's method are used. Aerodynamic forces are based upon the static approximation for the most part with aerodynamic damping added in certain cases. For the buckled plate, large deflection thin plate theory and static strip theory aerodynamics are used in a static stability analysis. It has been concluded for the unbuckled plate that the shear edge loading can have a drastic effect on flutter speeds. It also was found that the use of static aerodynamic forces can lead to spurious values for the flutter boundary in certain circumstances. The addition of aerodynamic damping helps to establish the correct-boundary. In the base of buckled plate, it has been concluded that the static stability boundary bears a significant relationship to the flutter boundary.

Nomenclature

a,b	=	plate dimensions in x and y directions, respectively
C_{rs}	=	coefficients of series expansions for unbuckled panel deflection
D	=	plate stiffness parameter, $Eh^3/12(1-\nu^2)$
E	=	Young's modulus for plate material
h	=	plate thickness
$k,\bar{k}$	=	frequency parameter, $\omega a^2(\rho_s h/D)^{1/2}$, and k/π^2, respectively
M	=	Mach number
N_x,N_y,N_{xy}	=	normal and shear edge loads per unit length (Fig. 1)
p	=	aerodynamic pressure
q	=	dynamic pressure, $\frac{1}{2}\rho_0 U^2$
Q	=	λ/π^4
r	=	aspect ratio, b/a
R_x,R_y,R_{xy}	=	$N_x a^2/D$, $N_y a^2/D$, and $N_{xy}a^2/D$, respectively
$\bar{R}_x,\bar{R}_y,\bar{R}_{xy}$	=	R_x/π^2, R_y/π^2, and R_{xy}/π^4, respectively
t	=	time

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* Associate Professor of Aeronautical and Astronautical Engineering.

† Research Specialist, Military Aircraft Systems Division.